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ABSTRACT

Let G be a $(2, m, n)$ -group and let x be the number of distinct primes dividing χ , the Euler characteristic of G . We prove, first, that, apart from a finite number of known exceptions, a non-abelian simple composition factor T of G is a finite group of Lie type with rank $n \leq x$. This result is proved using new results connecting the prime graph of T to the integer x .

We then study the particular cases $x = 1$ and $x = 2$. We give a general structure statement for $(2, m, n)$ -groups which have Euler characteristic a prime power, and we construct an infinite family of these objects. We also give a complete classification of those $(2, m, n)$ -groups which are almost simple and for which the Euler characteristic is a prime power (there are four such).

Finally we announce a result pertaining to those $(2, m, n)$ -groups which are almost simple and for which $|\chi|$ is a product of two prime powers. All such groups which are not isomorphic to $PSL_2(q)$ or $PGL_2(q)$ are completely classified.

Orientably regular maps with Euler characteristic divisible by few primes

Nick Gill

1. Introduction

Let m and n be positive integers. A $(2, m, n)$ -group is a triple (G, g, h) where G is a group, g (resp. h) is an element of G of order m (resp. n), and G has a presentation of form

$$\langle g, h \mid g^m = h^n = (gh)^2 = \cdots = 1 \rangle. \quad (1.1)$$

We will often abuse notation and simply refer to the group G as a $(2, m, n)$ -group. Clearly a group G is a $(2, m, n)$ -group if and only if it is a quotient of the group

$$\Gamma(m, n) = \langle \mathbf{g}, \mathbf{h} \mid \mathbf{g}^m = \mathbf{h}^n = (\mathbf{g}\mathbf{h})^2 = 1 \rangle$$

such that the images of \mathbf{g} and \mathbf{h} in G have orders m and n respectively.

It is well known that a $(2, m, n)$ -group G can be associated naturally with a map on an orientable surface \mathcal{S} ; this connection is fully explained in the beautiful paper of Jones and Singerman [JS78]. We note first that, if G is finite, then the surface \mathcal{S} is compact; furthermore in this case the group G has a natural regular action on the ‘half-edges’ of the associated map, and the map is as a result called *orientably regular* in the literature.

Suppose that (G, g, h) is a finite $(2, m, n)$ -group. Let us write E (resp. V , F) for the number of edges (resp. vertices, faces) of the map. Then G acts transitively on the set of edges (resp. set of vertices, set of faces) and the stabilizer in this action is cyclic of order 2 (resp. of order m , of order n). Now we can use these facts to calculate the Euler characteristic of the surface \mathcal{S} :

$$\chi = V - E + F = |G| \left(\frac{1}{m} - \frac{1}{2} + \frac{1}{n} \right) = -|G| \frac{mn - 2m - 2n}{2mn}. \quad (1.2)$$

It is well known that χ is an even integer and, moreover, that $\chi \leq 2$.

In this paper we investigate the situation where χ is divisible by few primes. We are interested in understanding the structure of the finite $(2, m, n)$ -group (G, g, h) in such a situation, particularly when G is non-solvable. The equation (1.2) implies that the quantity χ can be thought of as a property of the $(2, m, n)$ -group, as well as the surface, since we have an expression for χ in terms of $|G|, m$ and n . In what follows, then, we will refer to the *Euler characteristic of the $(2, m, n)$ -group (G, g, h)* and we will not consider the associated surface \mathcal{S} .

1.1. Results

In order to state our results we need a little notation: Fix a finite group K . A *subnormal* subgroup of K is a subgroup H for which there exists a chain H_1, \dots, H_k of subgroups of K such that $H \triangleleft H_1 \triangleleft \cdots \triangleleft H_k \triangleleft K$; a simple group J is a *composition factor* of K if there exist subnormal subgroups $H_1 \triangleleft H_2 \leq K$ such that $H_1/H_2 \cong J$. We can state our first theorem:

THEOREM 1.1. *Let x be a positive number. There exists a finite set S_x of finite simple groups, such that if T is a non-abelian composition factor of a finite $(2, m, n)$ -group whose Euler characteristic is divisible by exactly x distinct primes, then either T is a simple group of Lie type of rank at most x , or $T \in S_x$.*

By rank we mean the rank of the associated simple algebraic group; this number is equal to the number of nodes on the associated Dynkin diagram. Theorem 1.1 is stated in more detail and proved in §3 as Proposition 3.8. In particular, Proposition 3.8 supplies sufficient information so that, for each positive integer x , a set S_x can be enumerated.

The proof uses some simple ideas connected to the structure of Sylow subgroups (these have been used before to study regular maps; see for instance [BdNŠ05]), as well as properties of the prime graph of a group.

In §4 we study the structure of $(2, m, n)$ -groups G with Euler characteristic χ of form $\pm 2^a$ (note that, since χ is always even, this is the only possibility when χ is a prime power). In addition to a general structure statement for $(2, m, n)$ -groups of this form we give two additional results. First, we construct an infinite family of $(2, m, n)$ -groups with Euler characteristic $\chi = -2^a$ for any $a \equiv 24 \pmod{28}$; all of these have the particular property that $SL_2(8)$ is the only non-abelian composition factor. Secondly, we prove the following result concerning *almost simple* $(2, m, n)$ -groups (i.e. $(2, m, n)$ -groups (S, g, h) such that S is almost simple):

THEOREM 1.2. *Let (S, g, h) be an almost simple $(2, m, n)$ -group with Euler characteristic $\chi = \pm 2^a$ for some integer a . Then S is isomorphic to one of the following:*

- (a) $PSL_2(5)$ with $\{m, n\} = \{3, 5\}$ and $\chi = 2$;
- (b) $PSL_2(7)$ with $\{m, n\} = \{3, 7\}$ and $\chi = -4$;
- (c) $PGL_2(5)$ with $\{m, n\} = \{5, 6\}$ and $\chi = -16$;
- (d) $PGL_2(7)$ with $\{m, n\} = \{6, 7\}$ and $\chi = -64$.

What is more, in each case, for each pair (m, n) , there is a unique $(2, m, n)$ -group.

We clarify the meaning of the final sentence: Note first that there are four cases enumerated and, for each, there are two choices of pair (m, n) . The final sentence asserts that, in each case, each of the pairs (m, n) correspond to a unique $(2, m, n)$ -group up to isomorphism. (Two $(2, m, n)$ -groups (S_1, g_1, h_1) and (S_2, g_2, h_2) are *isomorphic as $(2, m, n)$ -groups* if there exists a group isomorphism $\phi : S_1 \rightarrow S_2$ such that $\phi(g_1) = g_2$ and $\phi(h_1) = h_2$.) In terms of surfaces this means that to each of the four enumerated cases there correspond precisely two orientably regular maps, and these maps are dual to each other.

In a forthcoming paper [Gil] we prove a result analogous to Theorem 1.2 for $(2, m, n)$ -groups which are almost simple and have Euler characteristic $\chi = -2^a s^b$ for some odd prime s . We announce that result here:

THEOREM 1.3. *Let (S, g, h) be an almost simple $(2, m, n)$ -group with Euler characteristic $\chi = -2^a s^b$ for some odd prime s and integers $a, b \geq 1$. Let T be the unique non-trivial normal subgroup in S . Then one of the following holds:*

- (a) $T = PSL_2(q)$ for some prime power $q \geq 5$ and either $S = T$ or $S = PGL_2(q)$ or else one of the possibilities listed in Table 1 holds.
- (b) $S = T$, $T.2$ or $T.3$, where T is a finite simple group and all possibilities are listed in Table 2.

What is more a $(2, m, n)$ -group exists in each case listed in Tables 1 and 2.

Full details concerning the groups listed in Tables 1 and 2 are given in [Gil]. For now we simply alert the reader to the fact that there may be more than one isomorphism class of abstract group associated with any given line of Tables 1 and 2.

The results of this paper suggest a number of avenues of future research. We discuss some of these more fully in the companion paper [Gil] however a couple of remarks are worth making here. Firstly observe that Theorems 1.2 and 1.3 suggest that a stronger version of Theorem 1.1 should hold under the extra assumption that G is almost simple; in particular we conjecture that in this case the conclusion should read T is a finite group of Lie type of rank n where $n \leq x$. Theorems 1.2 and 1.3 confirm this conjecture for $x \leq 2$.[†]

In a different direction we note that much of the work in this paper carries over to the study of groups associated with *non-orientable* regular maps. Indeed in this situation the structure of the group has more properties that we can exploit (for instance it is generated by three involutions) and we intend to address this question in a future paper.

[†]One could speculate further: for instance, under the suppositions of Theorem 1.1, is it true that if G is almost simple and T is a finite group of Lie type of rank $n = x - 1$, then, with finitely many exceptions, $G \leq \text{Inndiag}(T)$? (See [GLS98, Definition 2.5.10] for a definition of Inndiag .)

Group	$\{m, n\}$	χ
$S = PSL_2(9).2 \cong S_6$	$\{5, 6\}$	$-2^5 \cdot 3$
$S = PSL_2(9).(C_2 \times C_2)$	$\{4, 10\}$	$-2^3 \cdot 3^3$
$S = PSL_2(25).2$	$\{6, 13\}$	$-2^5 \cdot 5^3$

TABLE 1. Some $(2, m, n)$ -groups for which $\chi = -2^a s^b$

Group	$\{m, n\}$	χ
$T = SL_3(3)$	$\{4, 13\}$	$- S : T \cdot 2^2 \cdot 3^5$
$S = SL_3(3)$	$\{13, 13\}$	$-2^3 \cdot 3^5$
$S = SL_3(5)$	$\{3, 31\}$	$-2^4 \cdot 5^5$
$S = PSL_3(4).2$	$\{5, 14\}$	$-2^{10} \cdot 3^2$
$S = PSL_3(4).2$	$\{10, 7\}$	$-2^7 \cdot 3^4$
$S = PSL_3(4).3$	$\{15, 21\}$	$-2^5 \cdot 3^6$
$S = SU_3(3).2$	$\{4, 7\}$	$-2^4 \cdot 3^4$
$T = SU_3(3)$	$\{6, 7\}$	$- S : T \cdot 2^7 \cdot 3^2$
$S = SU_3(3)$	$\{7, 7\}$	$-2^4 \cdot 3^4$
$S = SU_3(4).2$	$\{6, 13\}$	$-2^8 \cdot 5^3$
$S = PSU_3(8)$	$\{7, 19\}$	$-2^8 \cdot 3^8$
$S = G_2(3).2$	$\{13, 14\}$	$-2^{12} \cdot 3^6$
$S = Sp_6(2)$	$\{7, 10\}$	$-2^9 \cdot 3^6$
$S = PSU_4(3).2$	$\{5, 14\}$	$-2^{11} \cdot 3^6$
$S = PSU_4(3).2$	$\{10, 7\}$	$-2^8 \cdot 3^8$
$S = SL_4(2).2 = S_8$	$\{10, 7\}$	$-2^7 \cdot 3^4$
$S = S_7$	$\{10, 7\}$	$-2^4 \cdot 3^4$
$S = A_9$	$\{10, 7\}$	$-2^6 \cdot 3^6$
$T = SU_4(2)$	$\{5, 6\}$	$- S : T \cdot 2^7 \cdot 3^3$
$S = SU(4, 2).2$	$\{10, 4\}$	$-2^5 \cdot 3^5$
$S = SU(4, 2).2$	$\{10, 5\}$	$-2^7 \cdot 3^4$
$S = SU(4, 2).2$	$\{10, 10\}$	$-2^6 \cdot 3^5$

TABLE 2. Some $(2, m, n)$ -groups for which $\chi = -2^a s^b$

1.2. The Literature

There is a substantial body of literature classifying regular and orientably regular maps in terms of the Euler characteristic χ of the associated surface. A complete classification (for orientable and non-orientable surfaces) has been obtained for values of χ between -2 and -200 [CD01, Con09]. When χ is prime, a complete classification is given in [BdNŠ05]; this is the first infinite family of surfaces for which a complete classification was obtained. This breakthrough result was followed by a complete classification when $\chi = -2p$ [ČŠT10], $\chi = -3p$ [CNŠ12] and $\chi = -p^2$ [CPŠ10] (where, in each case, p is a prime).

Our main results fit into this general scheme, however we do not give a complete classification of all orientably regular maps in each case but study instead the structure of the associated $(2, m, n)$ -group. Our focus on the case where the group is almost simple is deliberate, since these objects have a long history of study into which our work also fits.

This history has at its heart the question of which (almost) simple groups are $(2, 3, 7)$ -groups (otherwise known as *Hurwitz groups*), work on which is surveyed in [Con90]; notable results in this direction include those found in [Con87] for alternating groups, and in [Jon94] for Ree groups; we also mention [JS93] in which those $(2, 4, 5)$ -groups which are Suzuki groups are studied. In addition the question of which groups $PSL_2(q)$ and $PGL_2(q)$ are $(2, m, n)$ -groups has been studied in [Sah69]; similar questions are studied in [Mar10, CPŠ08].

More generally a result of Stein [Ste98] implies that all finite simple groups are $(2, m, n)$ -groups for some m and n (see also [MSW94] for a stronger statement). The same cannot be said of almost simple groups however: any almost simple group S with socle T such that S/T is non-cyclic of odd order, e.g. $PGL(3, 7^3)$, will fail to be a $(2, m, n)$ -group.

Additional motivation for considering almost simple $(2, m, n)$ -groups stems from recent work of Li and Širáň who seek to classify orientably regular maps acting quasiprimively on vertices [LŠ]. The main result of [LŠ] reduces the general problem of classifying all such objects to the problem of understanding a number of specific families, one of which is precisely the almost simple $(2, m, n)$ -groups.

1.3. Acknowledgments

I thank Jozef Širáň for introducing me to the study of orientably regular maps and for many very useful discussions on this subject. In addition my colleagues Robert Brignall and Ian Short of the Open University have participated in a reading group on this subject, out of which the current paper has grown.

I also wish to especially thank John Britnell who turned out to be the perfect person to help me understand automorphisms of simple groups. I am similarly indebted to Marston Conder, who stepped in when my muddle-headed approach to computer programming was threatening to send this paper into an infinite loop.

Finally I have been a frequent visitor to the University of Bristol during the period of research for this paper, and I wish to acknowledge the generous support of the Bristol mathematics department.

2. Background on groups

In this section we add to the notation already established, and we present a number of well-known results from group theory that will be useful in the sequel.

The following notation will hold for the rest of the paper: (G, g, h) is always a finite $(2, m, n)$ -group; (S, g, h) is always a finite almost simple $(2, m, n)$ -group; T is always a simple group. We use χ or χ_G to denote the Euler characteristic of the group G .

For groups H, K we write $H.K$ to denote an extension of H by K ; i.e. $H.K$ is a group with normal subgroup H such that $H.K/H \cong K$. In the particular situation where the extension is

split we write $H \rtimes K$, i.e. we have a semi-direct product. For an integer k we write H^k to mean $\underbrace{H \times \cdots \times H}_k$.

For an integer $n > 1$ we write C_n for the cyclic group of order n and D_n for the dihedral group of order n . We also sometimes write n when we mean C_n particularly when we are writing extensions of simple groups; so, for instance, $T.2$ is an extension of the simple group T by a cyclic group of order 2.

Let K be a group and let us consider some important normal subgroups. For primes p_1, \dots, p_k , write $O_{p_1, \dots, p_k}(K)$ for the largest normal subgroup of K with order equal to $p_1^{a_1} \cdots p_k^{a_k}$ for some non-negative integers a_1, \dots, a_k ; in particular $O_2(K)$ is the largest normal 2-group in K . We write $Z(K)$ for the centre of K and we write $F^*(K)$ for the *generalized Fitting subgroup* of K .

The generalized Fitting subgroup was defined originally by Bender [Ben70] and is fully explained in [Asc00]. Note that $F^*(K) = F(K)E(K)$ where $F(K)$ is the (classical) Fitting subgroup of K and $E(K)$ is the product of all quasisimple subnormal subgroups of K (a group L is *quasisimple* if it is perfect and $L/Z(L)$ is simple). We will need the fact that $C_K(F^*(K)) = Z(F^*(K))$ (recall that, if K is solvable, then $C_K(F(K)) = Z(F(K))$; this explains why $F^*(K)$ is called the *generalized Fitting subgroup*). Note that K is almost simple if and only if $F^*(K)$ is a finite non-abelian simple group.

Let a and b be positive integers. Write (a, b) for the greatest common divisor of a and b , and $[a, b]$ for the lowest common multiple of a and b ; observe that $ab = a, b$. For a prime p write a_p for the largest power of p that divides a ; write $a_{p'}$ for a/a_p . Write $\Phi_i(x)$ for the i -th cyclotomic polynomial. For fixed positive integers q and a with $a > 1$, we define a prime t to be a *primitive prime divisor* for $q^a - 1$ if t divides $\Phi_a(q)$ but t does not divide $\Phi_i(q)$ for any $i = 1, \dots, a - 1$. For fixed q we will write r_a to mean a primitive prime divisor for $q^a - 1$; then we can state (a version of) Zsigmondy's theorem [Zsi92]:

THEOREM 2.1. *Let q and a be positive integers with $a > 1$. Then there exists a primitive prime divisor r_a unless*

- (a) $(a, q) = (6, 2)$;
- (b) $a = 2$ and $q = 2^b - 1$ for some positive integer b .

Note that r_1 exists whenever $q > 2$; note too that, although r_2 does not always exist, still, for $q > 3$, there are always at least two primes dividing $q^2 - 1$. The following result is of similar ilk to Theorem 2.1; it is Mihăilescu's theorem [Mih04] proving the Catalan conjecture.

THEOREM 2.2. *Suppose that $q = p^a$ for some prime p and positive integer a . If $q = 2^a \pm 1$ and $q \neq p$, then $q = 9$.*

If T is a finite simple group of Lie type, then we use notation consistent with [GLS98, Definition 2.2.8] or, equivalently, with [KL90, Table 5.1.A]. In particular we write $T = T_n(q)$ to mean that T is of rank n , and q is a power of a prime p (in particular, for the Suzuki-Ree groups, we choose notation so that q is an integer). Using this definition, certain groups are excluded because they are non-simple, namely

$$A_1(2), A_1(3), {}^2A_2(2), {}^2B_2(2), C_2(2), {}^2F_4(2), G_2(2), {}^2G_2(3),$$

and $C_2(2), {}^2F_4(2), G_2(2), {}^2G_2(3)$ are replaced by their derived subgroups. A list of all isomorphisms between different groups of Lie type is given by [GLS98, Theorem 2.2.10] and [KL90, Proposition 2.9.1]. These isomorphisms imply that, for certain groups, the definition of *rank* is ambiguous; we

will take care to ensure that all results are stated so that they hold for such groups no matter what definition of rank is used.

When $T = T_n(q)$ is of Lie type we use [GLS98, (4.10.1)] to write the order of T as a product

$$\frac{1}{d} q^N \prod_i \Phi_i(q)^{n_i} \quad (2.1)$$

where d, N, n_i are non-negative integers and $\Phi_i(q)$ are cyclotomic polynomials as above. We will need criteria for when a Sylow t -subgroup of the simple group T is cyclic; the following result will be useful.

LEMMA 2.3. *Let $T = T_n(q)$ be a non-abelian simple group of Lie type with order given by (2.1). If an odd prime t divides $\Phi_i(q)$ and $n_i > 1$ then a Sylow t -subgroup of T is non-cyclic.*

Proof. Suppose first that T is not equal to either $A_3(q)$ or ${}^2A_3(q)$. Then [GLS98, Theorem 4.10.3] implies that a Sylow t -subgroup of T has t -rank greater than 1 and the result follows. If $T = A_3(q)$ (resp. ${}^2A_3(q)$) then the same conclusion holds unless $t = 3$ and t divides $q - 1$ (resp. $q + 1$); in this case we check the result directly and we are done. \square

We will want to apply the Lang-Steinberg theorem at several points in this paper. Let $T = T_n(q)$ be an untwisted group of Lie type; then T is the fixed set of a Frobenius endomorphism of a simple algebraic group $T_n(\overline{\mathbb{F}}_q)$. Suppose that ζ is a non-trivial field automorphism of T or, more generally, the product of a non-trivial field automorphism of T with a graph automorphism of T . Observe that ζ can be thought of as a restriction of an endomorphism of $T_n(\overline{\mathbb{F}}_q)$; what is more this endomorphism has the particular property that it has a finite number of fixed points. With the notation just established the Lang-Steinberg theorem implies the following result:

PROPOSITION 2.4. *Any conjugacy class of T which is stable under ζ (i.e. is stabilized set-wise) must intersect X non-trivially, where X is the centralizer in $T_n(q)$ of ζ .*

Proof. This is well known; see for instance [DM91, 3.10 and 3.12]. \square

For g an element of a group K write $o(g)$ for the order of g ; write g^K to mean the conjugacy class of g in K ; write $\text{Irr}(K)$ for the set of irreducible characters of K . The following proposition appears as an exercise in [Isa94, p. 45].

PROPOSITION 2.5. *Let g, h, z be elements of a group K . Define the integer*

$$a_{g,h,z} = \left| \{(x, y) \in g^K \times h^K \mid xy = z\} \right|.$$

Then

$$a_{g,h,z} = \frac{|K|}{|C_K(g)| \cdot |C_K(h)|} \sum_{\chi \in \text{Irr}(K)} \frac{\chi(g)\chi(h)\overline{\chi(z)}}{\chi(1)}.$$

3. Bounding the rank

In this section we prove (a stronger version of) Theorem 1.1. Recall that we assume that (G, g, h) is a $(2, m, n)$ -group with Euler characteristic χ ; we start with some lemmas concerning primes dividing χ .

LEMMA 3.1. *Suppose that t is an odd prime dividing $|G|$. If $|G|_t > |[m, n]|_t$, then t divides χ_G .*

Proof. Observe that (1.2) can be rewritten as follows:

$$\chi = -|G| \frac{mn - 2m - 2n}{2mn} = -\frac{|G|}{2[m, n]} \left(\frac{mn - 2m - 2n}{(m, n)} \right). \quad (3.1)$$

Clearly $\frac{mn - 2m - 2n}{(m, n)}$ is an integer and so, in particular, $\frac{|G|}{[m, n]}$ divides 2χ . The result follows. \square

An immediate corollary is the following result which is a particular case of Lemma 3.2 in [CPŠ10].

LEMMA 3.2. *Suppose that t is an odd prime divisor of $|G|$ such that a Sylow t -subgroup of G is not cyclic. Then $|G|_t > |[m, n]|_t$ and, in particular, t divides χ_G .*

Let N be a normal subgroup of G . Define m_N (resp. n_N) to be the order of gN (resp. hN) in G/N .

LEMMA 3.3. *Let N be a normal subgroup of the $(2, m, n)$ -group (G, g, h) . If G/N is not cyclic then $(G/N, gN, hN)$ is a $(2, m_N, n_N)$ -group.*

Proof. If $gh \in N$ then G/N is cyclic. If G/N is not cyclic, then $gh \notin N$ which implies that $(gN)(hN)$ is a non-trivial involution in G/N ; the result follows. \square

LEMMA 3.4. *Let N be a normal subgroup of G . If an odd prime t satisfies $|G/N|_t > |[m_N, n_N]|_t$ then t divides χ_G .*

Proof. Since t satisfies $|G/N|_t > |[m_N, n_N]|_t$ we have

$$|G|_t > |N|_t \cdot |[m_N, n_N]|_t.$$

This in turn implies that $|G|_t > |[m, n]|_t$ and Lemma 3.1 implies that t divides χ_G . \square

3.1. The prime graph

In this subsection we outline a technique that we can use to exploit Lemma 3.1.

Given a finite group K , let $\pi(K)$ be the set of all prime divisors of its order. Define the *prime graph*, or *Gruenberg-Kegel graph* $GK(K)$ as follows: the vertices of $GK(K)$ are elements of $\pi(K)$ and two vertices p, q are connected by an edge if and only if K contains an element of order pq . We will be interested in a subgraph of $GK(K)$ obtained by restricting $GK(K)$ to the set of primes for which the corresponding Sylow-subgroups of K are cyclic; we call this subgraph $GK_c(K)$; we also define $\pi_{nc}(K) \subseteq \pi(K)$ to be the set of primes for which the corresponding Sylow-subgroups of K are non-cyclic.

Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$; we recall some basic graph-theoretic definitions. A set of vertices of \mathcal{G} is called *independent* if its elements are pairwise non-adjacent; write $t(\mathcal{G})$ for the maximal number of vertices in independent sets of \mathcal{G} ; in graph theory $t(\mathcal{G})$ is usually called an *independence number* of the graph. A clique of the graph \mathcal{G} is a set of vertices in which every vertex is adjacent to every other vertex in the set (i.e. we have a copy of the complete graph on this set of vertices as a subgraph of \mathcal{G}). Write $m(\mathcal{G})$ for the minimum number of maximal cliques that contain all vertices of \mathcal{G} ; note the easy fact that $m(\mathcal{G}) \geq t(\mathcal{G})$.

The basic observation that underpins the following proposition is that the number of primes dividing χ is at least $m(\text{GK}(G)) - 3$, however by being careful we can slightly strengthen this.

PROPOSITION 3.5. *Let G be a finite $(2, m, n)$ -group of even order with corresponding Euler characteristic χ . Then the number of primes dividing $(\chi, |G|)$ is at least*

$$\max\{0, m(\text{GK}_c(G)) - 2\} + |\pi_{nc}(G)|.$$

The number of primes dividing $(\chi, |G|)$ is also at least $m(\text{GK}(G)) - 2$.

Proof. We start with the first assertion. Lemma 3.2 implies that if $t \in \pi_{nc}(G)$ then t divides χ . Now suppose that $t \in \pi(G) \setminus \pi_{nc}(G)$. If $t = 2$ then we know that $t | \chi$. If t is odd, then Lemma 3.1 implies that if $|G|_t > |[m, n]|_t$ then t divides χ . All of the primes dividing m lie in a maximal clique of $\text{GK}(G)$; similarly for n . If a Sylow 2-subgroup of G is non-cyclic then all of the vertices of $\text{GK}_c(G)$ correspond to odd primes. Hence there are at least $m(\text{GK}_c(G)) - 2$ distinct odd primes in $\pi(G) \setminus \pi_{nc}(G)$ that do not divide $[m, n]$ and the result follows.

Suppose now that a Sylow 2-subgroup of G is cyclic and non-trivial. Let $\text{GK}_{c,2'}(G)$ be the graph obtained by removing the vertex 2 and all edges adjacent to it. One can easily see that $m(\text{GK}_{c,2'}(G)) \geq m(\text{GK}_c(G)) - 1$. Clearly there at least $m(\text{GK}_{c,2'}(G)) - 2 \geq m(\text{GK}_c(G)) - 3$ distinct odd primes in $\pi(G) \setminus \pi_{nc}(G)$ that do not divide $[m, n]$; since 2 also divides $\chi(G)$ the first assertion follows.

The second assertion is similar: There are at least $m(\text{GK}(G)) - 3$ distinct odd primes that do not divide $[m, n]$; we know that 2 divides χ ; thus Lemma 3.1 implies that $m(\text{GK}(G)) - 2$ distinct primes divide $(\chi, |G|)$. \square

Note that if G is a finite $(2, m, n)$ -group of odd order then similar bounds hold (one must replace occurrences of “-2” with “-3” in the statement). In addition an easy variation of the argument used to prove Proposition 3.5 yields the following fact: if $\text{GK}(G)$ has an independent set of size s in which all primes are odd, then the number of primes dividing χ is at least $s - 1$.

In fact we can do better than Proposition 3.5 by considering normal subgroups N of G . For convenience we restrict to the situation where N has non-cyclic Sylow 2-subgroups; more general statements are possible but not needed. In particular Burnside’s theorem implies that the following result can be applied whenever N is non-solvable.

PROPOSITION 3.6. *Let (G, g, h) be a finite $(2, m, n)$ -group of even order and Euler characteristic χ . Let N be a normal subgroup of G with non-cyclic Sylow 2-subgroups. Then the number of primes dividing $\frac{|G|}{[m, n]}$ is at least*

$$\max\{0, m(\text{GK}_c(N)) - 2\} + |\pi_{nc}(N)|.$$

The number of primes dividing $\frac{|G|}{[m, n]}$ is also at least $m(\text{GK}(N)) - 2$.

Proof. We proceed as before, beginning with the first assertion. If $t \in \pi_{nc}(N)$ then $t \in \pi_{nc}(G)$ and Lemma 3.2 implies that then t divides χ . Now suppose that $t \in \pi(G) \setminus \pi_{nc}(G)$; by assumption t is odd. Observe that $m = o(g) = o(gN) \cdot o(n_g)$ where n_g is some element of N ; similarly $n = o(h) = o(hN) \cdot o(n_h)$ where n_h is some element of N . Since $o(gN)$ and $o(hN)$ divide $|G/N|$ and $|G| = |G/N| \cdot |N|$, we conclude that that if $|N|_t > |[o(n_g), o(n_h)]|_t$ then t divides $\frac{|G|}{[m, n]}$. Now there are at least $m(\text{GK}_c(N)) - 2$ distinct odd primes that do not divide $[o(n_g), o(n_h)]$ and the first assertion follows.

The second assertion is similar. \square

3.2. Applications to simple groups

We will be interested in combining Proposition 3.6 with results in the literature concerning the prime graph of a simple group. In particular an exhaustive study of independence sets and independence numbers in finite simple groups has been completed in [VV05]. Their results and Proposition 3.6 allow us to state the following two results which yield a stronger version of Theorem 1.1.

PROPOSITION 3.7. *Let $T = T_n(q)$ be a non-abelian simple group of Lie type of rank n . Let*

$$f(T) = \max\{0, m(\text{GK}_c(T)) - 2\} + |\pi_{nc}(T)|.$$

Then the following table gives lower bounds for $f(T)$.

T	Lower bound for $f(T)$
T is classical, $q \geq 4^\dagger$	n
${}^2B_2(q)$	2
${}^2G_2(q)$	3
${}^2F_4(q)$, $q \geq 8$	4
${}^2F_4(2)'$	3
${}^3D_4(q)$, $q \geq 4$	4
${}^3D_4(2)$ or ${}^3D_4(3)$	3
$G_2(q)$, $q \geq 4^\dagger$	3
$F_4(q)$, $q \geq 4^\dagger$	6
$E_6(q)$, $q \geq 4^\dagger$	8
${}^2E_6(q)$, $q \geq 4^\dagger$	8
$E_7(q)$, $q \geq 4^\dagger$	11
$E_8(q)$, $q \geq 4^\dagger$	15

\dagger In each of these cases the lower bound when $q = 3$ (resp. $q = 2$) is that for $q \geq 4$ reduced by 1 (resp. 2).

Proof. In the following proof we treat families of simple groups one at a time. For a simple group T we write the order of T in the form (2.1) and then apply Theorem 2.1 in association with Lemma 2.3 to give a lower bound on the number of distinct primes for which the Sylow subgroups of T are noncyclic; we then apply the results of [VV05] to the graph $\text{GK}_c(T)$ to obtain the desired lower bounds.

In what follows q is a power of a prime p . Recall that, for a positive integer a , we write r_a to denote a primitive prime divisor of $q^a - 1$, i.e. r_a is a prime which divides $\Phi_a(q)$ but does not divide $\Phi_i(q)$ for any $i = 1, \dots, a - 1$. We write t_1, t_2 for distinct primes dividing $q^2 - 1$; these exist whenever $q > 3$.

First let $T = A_n(q) \cong \text{PSL}_{n+1}(q)$. If $n = 1$ then $\pi_{nc}(T) = \{2\}$ and the result follows; if $n = 2, q > 2$ then $\pi_{nc}(T) \supseteq \{p, r_1\}$ and the result follows; if $(n, q) = (2, 2)$ then the result is trivial. Now assume that $n > 2$. If $q > 3$, then

$$\pi_{nc}(T) \supseteq \{p, t_1, t_2, r_3, \dots, r_{\lfloor \frac{n+1}{2} \rfloor}\}.$$

In addition we have an independence set in $GK(\pi_C(T))$ equal to

$$\{r_{\lfloor \frac{n+1}{2} \rfloor + 1}, \dots, r_n, r_{n+1}\}.$$

We conclude that $f(T) \geq n$ as required. If $q = 3$ then we have the same two sets but must account for the fact that t_2 does not exist; if $q = 2$ then we have the same two sets but must account for the fact that neither t_2 nor r_6 exist.

Let $T = {}^2A_n(q) \cong PSU_{n+1}(q)$ with $n > 1$; the situation is very similar to that of $A_n(q)$. If $n = 2$ and q is odd, then $\pi_{nc}(T) \supseteq \{2, p\}$ and the result follows; if $n = 2$ and $q > 2$ is even, then $\pi_{nc}(T) \supseteq \{2, r_2\}$ and the result follows; if $(n, q) = (2, 2)$ then T is not simple, a contradiction. Now assume that $n > 2$. If $q > 3$ then

$$\begin{aligned} \pi_{nc}(T) \supseteq & \{p, t_1, t_2\} \cup \{r_{i/2} \mid 6 \leq i \leq \lfloor \frac{n+1}{2} \rfloor, i \equiv 2 \pmod{4}\} \\ & \cup \{r_{2i} \mid 3 \leq i \leq \lfloor \frac{n+1}{2} \rfloor, i \equiv 1 \pmod{2}\} \\ & \cup \{r_i \mid 4 \leq i \leq \lfloor \frac{n+1}{2} \rfloor, i \equiv 0 \pmod{4}\}. \end{aligned}$$

In addition (see [VV05]) we have an independence set in $GK(\pi_C(T))$ equal to

$$\begin{aligned} & \{r_{i/2} \mid \lfloor \frac{n+1}{2} \rfloor < i \leq n+1, i \equiv 2 \pmod{4}\} \\ & \cup \{r_{2i} \mid \lfloor \frac{n+1}{2} \rfloor < i \leq n+1, i \equiv 1 \pmod{2}\} \\ & \cup \{r_i \mid \lfloor \frac{n+1}{2} \rfloor < i \leq n+1, i \equiv 0 \pmod{4}\} \end{aligned}$$

and the result follows. If $q = 3$ then, just as before, we have the same two sets but must account for the fact that t_2 does not exist; if $q = 2$ then we have the same two sets but must account for the fact that neither t_2 nor r_6 exist.

Let $T = B_n(q)$ or $C_n(q)$ with $n \geq 2$. Note that, by [GLS98, Theorem 2.2.10], we only need to consider $B_n(q) \cong P\Omega_{2n+1}(q)$ for q odd. If $q > 3$ then

$$\pi_{nc}(T) \supseteq \{p, t_1, t_2, r_4, r_6, \dots, r_{2\lfloor \frac{n}{2} \rfloor}\}.$$

In addition we have an independence set in $GK(\pi_C(T))$ equal to

$$\{r_{2\lfloor \frac{n}{2} \rfloor + 2}, \dots, r_{2n}\}$$

and the result follows. The same considerations as before for $q = 2, 3$ yield the result in these cases.

Let $T = D_n(q)$ with $n \geq 4$. If $q > 3$ then

$$\pi_{nc}(T) \supseteq \{p, t_1, t_2, r_4, r_6, \dots, r_{2\lfloor \frac{n}{2} \rfloor}\}.$$

In addition we have an independence set in $GK(\pi_C(T))$ equal to

$$\{r_{2i} \mid \lfloor \frac{n+1}{2} \rfloor < i < n\} \cup \{r_i \mid \lfloor \frac{n}{2} \rfloor < i \leq n, i \equiv 1 \pmod{2}\}$$

and the result follows. The same considerations as before for $q = 2, 3$ yield the result in these cases.

Let $T = {}^2D_n(q)$ with $n \geq 4$. If $q > 3$ then

$$\pi_{nc}(T) \supseteq \{p, t_1, t_2, r_4, r_6, \dots, r_{2\lfloor \frac{n-1}{2} \rfloor}\}.$$

In addition we have an independence set in $GK(\pi_C(T))$ equal to

$$\{r_{2i} \mid \lfloor \frac{n}{2} \rfloor < i \leq n\} \cup \{r_i \mid \lfloor \frac{n}{2} \rfloor < i < n, i \equiv 1 \pmod{2}\}$$

and the result follows. The same considerations as before for $q = 2, 3$ yield the result in these cases.

Now we consider the exceptional groups. If $T = {}^2B_2(q)$ then [VV05] implies that $t(GK(T)) = 4$; similarly if $T = {}^2G_2(q)$ then [VV05] implies that $t(GK(T)) = 5$; the result follows in each case.

If $T = {}^2F_4(q)$ with $q \geq 8$ then $\pi_{nc}(T) \supseteq \{p, r_1\}$ and [VV05] implies that we have an independence set in $GK(\pi_C(T))$ of size at least 4. If $T = {}^2F_4(2)'$ then [CCN⁺85] implies that $\pi_{nc}(T) = \{2, 3, 5\}$ and the result follows.

If $T = {}^3D_4(q)$ with $q > 3$ then [Kle88] implies that $\pi_{nc}(T) \supseteq \{p, t_1, t_2\}$ and we have an independence set in $GK(\pi_C(T))$ equal to $\{r_{12}, r_3, r_6\}$. If $q = 3$ then we have the same two sets but must account for the fact that t_2 does not exist; if $q = 2$ then [CCN⁺85] implies that $\pi_{nc}(T) = \{2, 3, 7\}$ and the result follows in each case.

If $T = G_2(q)$ with $q \geq 4$ then $\pi_{nc}(T) \supseteq \{p, t_1, t_2\}$ and [VV05] implies that we have an independence set in $GK(\pi_C(T))$ equal to $\{r_3, r_6\}$. If $T = G_2(3)$ then we have the same two sets but must account for the fact that t_2 does not exist. (Recall that $G_2(2)$ is not simple.)

If $T = F_4(q)$, $E_6(q)$, ${}^2E_6(q)$ or $E_7(q)$ $\pi_{nc}(T) \supseteq \{p, t_1, t_2, r_3, r_4, r_6\}$. Making the usual adjustments for $q = 2, 3$ we obtain the result for $T = F_4(q)$. Using [VV05] the results follow in the other cases also.

If $T = E_8(q)$ then $\pi_{nc}(T) \supseteq \{p, t_1, t_2, r_3, r_4, r_6, r_8, r_{10}, r_{12}\}$; now [VV05] implies that we have an independence set in $GK(\pi(T))$ equal to $\{r_7, r_9, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\}$ and the result follows. \square

PROPOSITION 3.8. *Let (G, g, h) be a finite $(2, m, n)$ -group with corresponding Euler characteristic χ . Let T be a non-abelian simple group which is a composition factor of G . Write x for the number of distinct primes dividing $(\chi, |T|)$.*

- (a) *If $T = A_n$, the alternating group on n -letters, then $n < 2p_{x+1}$ where p_x is the x -th smallest prime;*
- (b) *If $T = T_n(q)$ is a finite simple group of Lie type of rank n , then $n \leq x$ or $n \leq x + 2$ and $q \leq 3$.*

Before we prove this result, a definition: a group J is a *chief factor* for a group K if there exist subgroups H_1, H_2 both normal in K such that $H_2/H_1 \cong J$, and there is no normal subgroup H_3 of K such that $H_1 < H_3 < H_2$.

Proof. Since T is a composition factor of G we conclude that T^k is a chief factor of K for some $k \geq 1$. Let H_1, H_2 be normal subgroups of G such that $H_1 < H_2$ and $H_2/H_1 \cong T^k$. If $|G/H_1|_t > |[o(gH_1), o(hH_1)]|_t$ then $|G|_t > [o(g), o(h)]_t$ and so t divides χ . Thus it is sufficient to assume that H_1 is trivial and prove that $|G|_t > [o(g), o(h)]_t$, i.e. $|G|_t > [m, n]_t$.

Suppose first that $T \cong A_n$. If $k = 1$ then we use the fact that, for a prime $t < \frac{n}{2}$, the Sylow t -subgroup of A_n is non-cyclic; thus $|\pi_{nc}(T)| > x$ for $n \geq 2p_{x+1}$. Now Proposition 3.6, applied with $N = H_2 \cong T$ gives the result. If $k > 1$ then all Sylow t -subgroups of $H_2 \cong T^k$ are non-cyclic. Thus $|\pi_{nc}(H_1)| = |\pi(T)| > x$ for $n \geq p_x$, and Proposition 3.6 gives the result once again.

Now suppose that T is a finite simple group of Lie type of rank n ; define $f(T)$ as in Proposition 3.7 and observe that, for $q \geq 4$, $f(T) \geq n$. Now Proposition 3.6, applied with $N = H_2 \cong T$ implies that $x \geq f(T)$ and the result follows. If $k > 1$ then, again, all Sylow t -subgroups of $H_2 \cong T^k$ are non-cyclic. Now it is a triviality that $|\pi(T)| \geq f(T)$ and so once again, provided $q \geq 4$, we have $x \geq |\pi(T)| \geq n$ and the result follows. \square

There are obvious ways to improve Proposition 3.8: as the proof suggests, one can give much stronger bounds whenever G has a non-simple non-abelian chief factor. As well, for particular families of finite simple groups of Lie type, we may conclude stronger bounds for the value $f(T)$ than those listed in Proposition 3.7. Finally we note that, by considering the prime graph of a simple group T , we lose information about, for instance, elements of order a product of prime

powers (rather than products of primes); this could be rectified by studying the *spectrum* of the group T , i.e. the maximal elements in the poset of element orders of the group.

4. Euler characteristic $\pm 2^a$

In this section we study the structure of the $(2, m, n)$ -group G under the added assumption that $\chi = \pm 2^a$ for some integer a . The first couple of results are independent of the classification of finite simple groups.

PROPOSITION 4.1. *Let G be a non-solvable finite $(2, m, n)$ -group with Euler characteristic $\chi = 2^a$ for some positive integer 2 . Write $\overline{G} = G/O_2(G)$. Then \overline{G} has a normal subgroup isomorphic to $M \times T_1 \times \cdots \times T_k$ where $F^*(M)$ is cyclic of odd order, k is a positive integer, T_1, \dots, T_k are simple groups such that, for all $i \neq j$, $(|T_i|, |T_j|)$ is a power of 2, and $\overline{G}/(M \times T_1 \cdots T_k)$ is isomorphic to a subgroup of $\text{Out}(T_1 \times \cdots \times T_k)$.*

Proof. Recall that $F^*(\overline{G}) = F(\overline{G})E(\overline{G})$ where $F(\overline{G})$ is nilpotent, $E(\overline{G})$ is a product of quasisimple groups, and both are normal in \overline{G} . Since $\overline{G} = G/O_2(G)$ we conclude that $F(\overline{G})$ has odd order.

Suppose that there exists an odd prime t such that the Sylow t -subgroups of $F(\overline{G})$ or $E(\overline{G})$ are non-cyclic. In particular the Sylow t -subgroups of G are non-cyclic, and so Lemma 3.2 implies that t divides χ which is a contradiction. We conclude, first, that $F(\overline{G})$ is cyclic.

Suppose, next, that $E(\overline{G})$ is trivial. Then $F^*(G) = F(G)$ is cyclic. Since $C_G(F^*(G)) = Z(F^*(G))$ this implies that $G/F^*(G)$ is isomorphic to a subgroup of the automorphism group of a cyclic group; in particular $G/F^*(G)$ is solvable. Since $F^*(G)$ is solvable we conclude that G is solvable, a contradiction. Thus $E(\overline{G})$ is non-trivial. What is more, since we know that the Sylow T -subgroups of $E(\overline{G})$ are non-cyclic, we conclude that $E(\overline{G}) = E_1 \times \cdots \times E_k$ for some quasisimple groups E_1, \dots, E_k such that, for all $i \neq j$, $(|E_i|, |E_j|)$ is a power of 2.

A result of Zassenhaus [Zas36, Theorem 15] says that no perfect group with cyclic q -Sylow subgroups can contain a normal subgroup of order q ; we conclude, therefore, that, for $i = 1, \dots, k$, $Z(E_i)$ is trivial or has even order. In the latter case, since $Z(E_i)$ is normal in \overline{G} , we have a contradiction. We conclude that, for $i = 1, \dots, k$, $E_i = T_i$, a simple group.

Now let $M = C_{\overline{G}}(T_1 \times \cdots \times T_k)$; since $T_1 \times \cdots \times T_k$ is normal in \overline{G} , so is M ; what is more $F^*(M)$ is cyclic of odd order. Now, since $\overline{G}/(M \times T_1 \times \cdots \times T_k)$ is isomorphic to a subgroup of $\text{Out}(T_1 \times \cdots \times T_k)$, we are done. \square

We will strengthen Proposition 4.1 by studying the groups T and M in turn, and then applying Lemma 3.4.

Let us recall some basic facts about odd-order cyclic p -groups. Suppose that C is cyclic of order p^a for some odd prime p and positive integer a . Then $\text{Out} C$ is cyclic of order $p^{a-1}(p-1)$. Let D be the unique subgroup of $\text{Out} C$ of order $p-1$ and consider the group $C \rtimes D$. Take $g = (g_c, g_d) \in C \rtimes D$ and suppose that g_d has order $k > 1$. Then, for $i \leq k$,

$$g^i = (g_c, g_d)^i = (g_c \cdot g_c^{g_d} \cdot g_c^{g_d^2} \cdots g_c^{g_d^{i-1}}, g_d^i).$$

Now for $i < k$ we have $g_d^i \neq 1$. On the other hand for $i = k$ observe that $g_c \cdot g_c^{g_d} \cdot g_c^{g_d^2} \cdots g_c^{g_d^{k-1}}$ is fixed by g_d in the action of D on C . Since D acts fixed-point-freely on C (i.e. no non-trivial element of D fixes a non-trivial element of C) we conclude that the order of g is k , the order of g_d .

The facts just described will be useful as we prove the following result concerning the group M .

PROPOSITION 4.2. *Let (M, g, h) be a solvable finite $(2, m, n)$ -group with Euler characteristic χ such that $\frac{|M|}{[m, n]} = \pm 2^a$. Write $\overline{M} = M/O_2(M)$. Then \overline{M} has a normal subgroup C such that C is cyclic and \overline{M}/C is isomorphic to $\{1\}$, C_2 or $C_2 \times C_2$.*

Proof. Lemmas 3.2 and 3.4 imply that all odd order Sylow subgroups of M are cyclic. Thus, in particular, $F^*(\overline{M})$ is cyclic; write $F^*(\overline{M}) = C_1 \times C_2 \times \cdots \times C_k$ where C_i is cyclic of order $p_i^{a_i}$ for some odd prime p_i and positive integer a_i . Then \overline{M} is isomorphic to a subgroup of $(C_1 \rtimes \text{Out}C_1) \times \cdots \times (C_k \rtimes \text{Out}C_k)$.

Consider the projection \overline{M}_1 of \overline{M} onto the first coordinate of this direct product and write g_1, h_1 for the images of g and h in \overline{M}_1 , with m_1 and n_1 their respective orders. Then \overline{M}_1 is isomorphic to a subgroup of $C_1 \rtimes \text{Out}C_1$ and, by Lemmas 3.3 and 3.4, \overline{M}_1 is cyclic or is a $(2, m_1, n_1)$ -group such that $|M_1|_{p_1} = [m_1, n_1]_{p_1}$. If \overline{M}_1 is cyclic then we conclude that $\overline{M}_1 = C_1$.

Suppose that \overline{M}_1 is not cyclic. Now $|\text{Out}C_1| = p_1^{a_1-1}(p_1 - 1)$ and, since all odd order Sylow subgroups of M are cyclic, we conclude that \overline{M}_1 is isomorphic to a subgroup of $C_1 \rtimes D_1$ where D_1 is the subgroup of automorphisms of C_1 of order $p_1 - 1$.

As described above the only elements of \overline{M}_1 of order divisible by p_1 lie in the normal subgroup isomorphic to C_1 . Since $|\overline{M}_1|_{p_1} = [m_1, n_1]_{p_1}$ we conclude that either g_1 or h_1 has order $p_1^{a_1}$ and generates a normal p_1 -subgroup in \overline{M}_1 . Furthermore, since $g_1 h_1$ has order 2, the discussion above implies that either g_1 or h_1 has order 2. We conclude that \overline{M}_1 is isomorphic to $D_{2p_1^{a_1}}$, the dihedral group of order $2p_1^{a_1}$.

The same consideration can be applied to projections onto the remaining coordinates to obtain that, in every case, the image is cyclic or dihedral. We conclude that

$$C_1 \times \cdots \times C_k \triangleleft \overline{M} \lesssim (C_1 \rtimes 2) \times \cdots \times (C_k \rtimes 2).$$

Observe that the quotient of \overline{M} by the normal subgroup $C_1 \times \cdots \times C_k$ is clearly an elementary abelian 2-group. Now, since $M/(C_1 \times \cdots \times C_k)$ is isomorphic to the quotient of a $(2, m, n)$ -group (and hence, in particular, is generated by at most two elements) we conclude that it is of order 1, 2, or 4. \square

Next we list the possible isomorphism class of T in Proposition 4.1.

PROPOSITION 4.3. *Let (S, g, h) be a non-abelian almost simple finite $(2, m, n)$ -group with Euler characteristic χ and suppose that $\frac{|S|}{[m, n]} = \pm 2^a$ for some integer a . Then S is isomorphic to one of the following:*

- (a) $PSL_2(p)$ for $p \geq 5$ an odd prime equal to $2^a \pm 1$ for some integer a ;
- (b) $PGL_2(p)$ for $p \geq 5$ an odd prime equal to $2^a \pm 1$ for some integer a ;
- (c) $SL_2(2^a)$ for $a \geq 3$ an integer;

Proof. Write $T = F^*(S)$. We must first show that T is of Lie type and isomorphic to $A_1(q)$ for some $q \geq 5$.

Suppose first that T is sporadic; then, using [CCN⁺85], we see that $\pi_{NC}(T) \geq 2$, a contradiction with Proposition 3.6. Next suppose that $T \cong A_n$ is alternating; if $n \geq 6$ then T has non-cyclic Sylow t -subgroups for $t = 2$ and 3, which contradicts Proposition 3.6. If $n \leq 4$ then T is not simple, a contradiction, and we are left with $n = 5$. But in this case $T \cong PSL_2(5)$ as listed.

Thus we need only consider the situation when T is of Lie type. Referring to Proposition 3.8 we immediately conclude that T has rank at most 1 or else $q \leq 3$. In fact, by referring to Propositions 3.6 and 3.7 we can give an explicit list of all possibilities that we need to consider. In addition to

$A_1(q)$ we must rule out

$$A_2(2), {}^2A_2(2), G_2(2), A_2(3), A_3(2), C_2(2)', C_2(3), C_3(2), {}^2A_2(3), {}^2A_3(2). \quad (4.1)$$

Observe first that $A_2(2) \cong A_1(7)$ as required; next observe that ${}^2A_2(2), G_2(2)$ are not simple and can be excluded; now the remaining groups all have non-cyclic Sylow t -subgroups for $t = 2$ and 3 , contradicting Proposition 3.6. We conclude that $T \cong A_1(q) \cong PSL_2(q)$ for some prime power $q \geq 5$.

Suppose first that q is odd. If q is not equal to p then T has non-cyclic Sylow t -subgroups for $t = 2$ and p , contradicting Proposition 3.6. Thus we assume that $q = p$ and so $S = PSL_2(p)$ or $PGL_2(p)$. Suppose $S = PSL_2(p)$; the maximal orders of elements in $PSL_2(p)$ are p , $\frac{p-1}{2}$ and $\frac{p+1}{2}$; in order for mn to be divisible by all odd primes dividing these three numbers we must have one of them equal to a power of 2. Similarly, if $S = PGL_2(p)$, then the maximal orders of elements are $p, p-1$ and $p+1$ and the same condition holds.

Suppose next that $q = 2^a$ for some integer $a \geq 3$; we must show that $S = T$. Suppose not; then $S = T \rtimes C_f$ where f divides a and C_f is a non-trivial cyclic group generated by a field automorphism δ . Maximal orders of elements in T are $2, 2^a + 1$ and $2^a - 1$ hence $2^a - 1$ must divide one of the elements in $\{m, n\}$ and $2^a + 1$ must divide the other.

For $x \in PSL_2(q)$, consider $(x, \delta) \in PSL_2(q) \rtimes \langle \delta \rangle$ and observe that

$$y = (x, \delta)^f = x \cdot x^{\delta^{f-1}} \cdots x^{\delta^2} \cdot x^\delta.$$

In particular observe that $y^\delta = x^\delta \cdot y \cdot x^{-\delta}$ and we obtain that y lies in a conjugacy class of $PSL_2(q)$ that is stable under δ . Now we apply Proposition 2.4 and conclude that y is conjugate in $PSL_2(q)$ to an element of $PSL_2(q_0)$ where $q = q_0^f$. In particular the element (x, δ) has order at most $f(q_0 + 1)$.

Now, since g and h generate S , we know that (using semidirect product notation) one of them must be equal to (x, δ) for some $x \in PSL_2(q)$. We conclude that $q - 1 \leq f(q_0 + 1)$ and so $q \leq 8$. Then $q = 8$ and $S = T.3$ and, since g and h generate S and multiply to give an involution, we have $o(gT) = o(hT) = 3$. We conclude that the order of both g and h divides $3|PSL_2(2)|$; but now $\frac{|S|}{|m, n|}$ is divisible by 7, a contradiction. \square

Let us gather together all the results of this section so far.

COROLLARY 4.4. *Let (G, g, h) be a finite $(2, m, n)$ -group with Euler characteristic $\chi = \pm 2^a$ for some positive integer 2. Write $\overline{G} = G/O_2(G)$. Then \overline{G} has a normal subgroup N such that*

- (a) \overline{G}/N is isomorphic to a subgroup of $C_2 \times C_2$;
- (b) $N \cong C \times T$ where C is cyclic and $(|C|, |T|) = 1$;
- (c) T is trivial or $T \cong SL_2(2^a)$ for some $a \geq 3$ or $T \cong PSL_2(p)$ where $p \geq 5$ is an odd prime equal to $2^a \pm 1$ for some integer a .

Proof. We may assume that G is non-solvable since Proposition 4.2 implies the result in the solvable case. Then Proposition 4.1 implies that $F^*(\overline{G}) \cong C \times T_1 \times \cdots \times T_k$ where C is cyclic, T_1, \dots, T_k are simple groups such that, for all $i \neq j$, $(|T_i|, |T_j|)$ is a power of 2, and $(|C|, |T|) = 1$.

The fact that $(|T_i|, |T_j|)$ is a power of 2 for all $i \neq j$ implies that $T_i \not\cong T_j$ for all $i \neq j$. This implies, in particular, that each T_i is a normal subgroup in \overline{G} . Let V_i be the centralizer of T_i in \overline{G} ; then \overline{G}/V_i is an almost simple group with a normal subgroup isomorphic to T_i . Let m_i (resp. n_i) be the order of the element $(gO_2(G))V_i \in \overline{G}/V_i$ (resp. $(hO_2(G))V_i \in \overline{G}/V_i$). By Lemma 3.4 we know that $\frac{|\overline{G}/V_i|}{|m_i, n_i|}$ is a power of 2. Then Proposition 4.3 gives the possible isomorphism type of T_i . Observe that all of the simple groups listed in Proposition 4.3 have order divisible by 3. We conclude that $k = 1$ and we write $F^*(\overline{G}) = C \times T$ where T is isomorphic to one of the simple groups listed in Proposition 4.3.

Suppose that \overline{G}/T is cyclic. Since $\text{Out}(T)$ has order at most 2, we know that $V = C_{\overline{G}}(T)$ is a normal subgroup of index at most 2 in \overline{G} . Thus V is cyclic of odd order and we conclude that $C = V$ and the result follows.

Suppose instead that \overline{G}/T is a solvable $(2, m, n)$ -group. Write m_T (resp. n_T) for the order of the element $(gO_2(G))T \in \overline{G}/T$ (resp. $(hO_2(G))T \in \overline{G}/T$). Then, by Lemma 3.4, we know that $\frac{|\overline{G}/T|}{[m_T, n_T]}$ is a power of 2. Then Proposition 4.2 implies that \overline{G}/T has a normal odd order cyclic subgroup X with quotient an elementary abelian 2-group of order at most 4. Again, since $\text{Out}(T)$ has order at most 2, we conclude that the centralizer of T in \overline{G} has a normal odd-order subgroup of index dividing 4; thus, this is C . The quotient $\overline{G}/(C \times T)$ is an elementary abelian 2-group of order at most 4, and the result follows. \square

4.1. Almost simple groups and an infinite family

To some extent Cor. 4.4 reduces the question of classifying regular maps with Euler characteristic a power of 2 to a number theoretic question. Let us consider the problem of classifying such maps with corresponding groups G such that $O_2(G) = 1$. We can analyse the structure of G one case at a time.

Suppose, for instance, that G has a normal subgroup $C \times T$ where $T \cong SL_2(2^a)$ for some $a \geq 3$, C cyclic of odd order. Write $|C| = p_1^{a_1} \cdots p_k^{a_k}$ where p_1, \dots, p_k are distinct primes. If $G = C \times T$, then it is an easy matter to see that $\{m, n\} = \{|C|(2^a - 1), |C|(2^a + 1)\}$ and the Euler characteristic can be calculated explicitly.

Similarly if $|G : (C \times T)| = 2$ then, up to a reordering of primes and for some $i \in \{1, \dots, n-1\}$, we have

$$\{m, n\} \in \{\{2p_1^{a_1} \cdots p_i^{a_i}(2^a - 1), |C|(2^a + 1)\}, \{2p_1^{a_1} \cdots p_i^{a_i}(2^a + 1), |C|(2^a - 1)\}\}.$$

Again the Euler characteristic can be calculated explicitly.

In this section we give two results of the kind just described. First of all we completely classify the almost simple $(2, m, n)$ -groups with Euler characteristic a power of 2, thereby proving Theorem 1.2; this is the substance of Propositions 4.5 and 4.6. Furthermore we study the particular case when $T \cong SL_2(8)$ and prove, in Proposition 4.7, that there are an infinite number of such maps.

PROPOSITION 4.5. *Let (S, g, h) be an almost simple finite $(2, m, n)$ -group with Euler characteristic χ and suppose that $\chi = \pm 2^a$ for some integer a . Then S is isomorphic to one of the following:*

- (a) $PSL_2(5)$ with $\{m, n\} = \{3, 5\}$ and $\chi = 2$;
- (b) $PSL_2(7)$ with $\{m, n\} = \{3, 7\}$ and $\chi = -4$;
- (c) $PGL_2(5)$ with $\{m, n\} = \{5, 6\}$ and $\chi = -16$;
- (d) $PGL_2(7)$ with $\{m, n\} = \{6, 7\}$ and $\chi = -64$.

Proof. Note first that, since $\chi = \pm 2^a$, and since (3.1) implies that $|S|/[m, n]$ divides 2χ , we conclude that $|S|/[m, n]$ is a power of 2. Now we go through the possibilities given by Proposition 4.3 and calculate the Euler characteristic.

Suppose first that $S = SL_2(q)$ with $q = 2^a$ for $a \geq 3$. Then we must have $\{m, n\} = \{p-1, p+1\}$ and (1.2) implies that

$$\begin{aligned} \chi &= q(q+1)(q-1) \left(\frac{1}{q+1} + \frac{1}{q-1} - \frac{1}{2} \right) \\ &= -\frac{1}{2}q(q^2 - 4q + 1). \end{aligned}$$

Now $q^2 - 4q + 1$ is odd, therefore we require that $q^2 - 4q + 1 = 1$ and thus $q = 4$ which is a contradiction. (Note that $SL_2(4)$ **does** give an example of a regular map, which we shall see when we examine the isomorphic group $PSL_2(5)$.)

Suppose next that $S = PSL_2(p)$ with $p = 2^a \pm 1$ a prime at least 5. Suppose first that $p = 2^a + 1$; then we must have $\{m, n\} = \{p, \frac{p+1}{2}\}$ and (1.2) implies that

$$\begin{aligned}\chi &= \frac{1}{2}p(p+1)(p-1) \left(\frac{2}{p+1} + \frac{1}{p} - \frac{1}{2} \right) \\ &= -\frac{1}{4}(p-1)(p^2 - 5p - 2).\end{aligned}\tag{4.2}$$

Since $p \equiv 1 \pmod{2a}$ we have that $p^2 - 5p - 2 \equiv -6 \pmod{2^a}$ and we conclude that $p^2 - 5p - 2 = 2$ and so $p = 5$ as required.

Suppose next that $p = 2^a - 1$; then we must have $\{m, n\} = \{p, \frac{p-1}{2}\}$ and (1.2) implies that

$$\begin{aligned}\chi &= \frac{1}{2}p(p+1)(p-1) \left(\frac{2}{p-1} + \frac{1}{p} - \frac{1}{2} \right) \\ &= -\frac{1}{4}(p+1)(p^2 - 7p + 2).\end{aligned}\tag{4.3}$$

Since $p \equiv -1 \pmod{2a}$ we have that $p^2 - 7p + 2 \equiv 10 \pmod{2^a}$ and we conclude that $p^2 - 7p + 2 = 2$ and so $p = 7$ as required.

We are left with the possibility that S is not simple, but has $F^*(S) = PSL_2(p)$; then $S = PGL_2(p)$. Since $p = 2^a \pm 1$ we know that $[m, n]$ must be divisible by $p^{\frac{p \pm 1}{2}}$. The only elements with order divisible by p are of order p and lie inside $PSL_2(p)$; thus, since we need two elements that generate $PGL_2(p)$, the remaining element must lie outside $PSL_2(p)$ and have order divisible by $\frac{p \pm 1}{2}$; this implies that the element has order $p \pm 1$.

When $p = 2^a + 1$ we find that

$$\begin{aligned}\chi &= p(p+1)(p-1) \left(\frac{1}{p+1} + \frac{1}{p} - \frac{1}{2} \right) \\ &= -\frac{1}{2}(p-1)(p^2 - 3p - 2).\end{aligned}\tag{4.4}$$

Since $p \equiv 1 \pmod{2a}$ we have that $p^2 - 3p - 2 \equiv -4 \pmod{2^a}$ and we conclude that $2^a = 4$ and so $p = 5$ as required.

When $p = 2^a - 1$ we find that

$$\begin{aligned}\chi &= p(p+1)(p-1) \left(\frac{1}{p-1} + \frac{1}{p} - \frac{1}{2} \right) \\ &= -\frac{1}{2}(p+1)(p^2 - 5p + 2).\end{aligned}\tag{4.5}$$

Since $p \equiv -1 \pmod{2a}$ we have that $p^2 - 5p + 2 \equiv 8 \pmod{2^a}$ and we conclude that $2^a = 8$ and so $p = 7$ as required. \square

We need to check that the four examples listed in Proposition 4.5 really do occur; in the next proposition we do this and, moreover, we show that, in each case, the listed $(2, m, n)$ -group is unique.

PROPOSITION 4.6. *Let S be one of the four groups listed in Proposition 4.5, and let m and n be the listed integers in increasing order. Then there exist $g, h \in S$ such that (S, g, h) is a $(2, m, n)$ -group. Furthermore if (S, g', h') is a $(2, m, n)$ -group then there exists a group automorphism $\phi : S \rightarrow S$ such that $\phi(g) = g'$ and $\phi(h) = h'$.*

Proof. Suppose first that $S = PSL_2(q)$ with $q = 5$ or 7 . We consult [CCN⁺85] to obtain the maximal subgroups of S and observe first that any pair of elements $g, h \in S$ of orders $m = 3$ (resp. $n = q$) must generate S . Next we observe that, for any triple of elements (g, h, z) of orders $(3, q, 2)$, all non-trivial characters take a zero value at g, h or z . This fact along with Proposition 2.5 implies that

$$a_{g,h,z} = |\{(x, y) \in g^S \times h^S \mid xy = z\}| = \frac{|S|}{|C_S(g)| \cdot |C_S(h)|} = \frac{|S|}{3 \cdot q} = \frac{q^2 - 1}{6}. \quad (4.6)$$

Thus $a_{g,h,z} > 0$ and we can choose g and h so that the order of g (resp. h, gh) is m (resp. $n, 2$), as required. Assume from here on that g and h have these properties.

Suppose next that $g' \in g^S$ and $h' \in h^S$ satisfy $(gh)^2 = 1$. Since there is only one class of involutions in S we can conjugate g' and h' to ensure that $g'h' = gh$. Next observe that, for both $q = 5$ and $q = 7$, the right hand side of (4.6) has size equal to $|C_S(gh)|$. Now $C_S(gh) \cap C_S(g)$ is trivial and so pairs (g^x, h^x) are all distinct for $x \in C_S(gh)$. Since $g^x h^x = gh$ in every case, these are all possible pairs in (g^S, h^S) which multiply to give z ; we conclude that there exists $x \in S$ such that $g' = g^x$ and $h' = h^x$.

Finally observe that, in both cases, S contains a unique conjugacy class of order 3 and two of order q . We must show that if (S, g, h) and (S, g, h') are $(2, m, n)$ -groups with $h \notin h^S$ then there exists a group automorphism $\phi : S \rightarrow S$ such that $\phi(h) = h'$. The two conjugacy classes of order q are fused in $S.2 = PGL_2(q)$. Thus we set f to be an element of order 6 in $S.2$ such that $f^2 = g$. Define $\phi_0 : S \rightarrow S, s \mapsto s^f$; then, by assumption, $h^f \in (h')^S$. If $h^f = h'$ then take $\phi = \phi_0$ and we are done. Otherwise choose f_1 such that $(h^{ff_1}) = h'$ and $g^{f_1} = g$ (this is possible by the previous paragraph) and set $\phi(s) = s^{ff_1}$ and we are done.

Now suppose that $S = PGL_2(q)$ with $q = 5$ or 7 . We consult [CCN⁺85] to obtain the maximal subgroups of S and observe first that any pair of elements $g, h \in S$ of orders m (resp. n) must generate S . We observe, furthermore, that exactly one of the elements g, h lies in $T = PSL_2(q)$; this implies, in particular, that $gh \notin PSL_2(q)$.

There is a unique conjugacy class of involutions in $S \setminus T$ to which gh must belong; similarly there is a unique conjugacy class of order m (resp. n) hence the conjugacy class containing g (resp. h) is completely determined. Choose z in the conjugacy class of involutions in $S \setminus T$. We calculate $a_{g,h,z}$ using Proposition 2.5 and in both cases obtain $a_{g,h,z} = 12$. In particular we conclude that we can choose g, h so that gh is an involution and (S, g, h) is a $(2, m, n)$ -group. Assume that g and h are chosen in this way from here on.

Now suppose that g' (resp. h') are elements such that (S, g', h') is also a $(2, m, n)$ -group. We know that $g'h'$ lies in the same conjugacy class as gh ; thus, by conjugating appropriately, we may assume that $g'h' = gh$. Since $a_{g,h,z} = 12$ we know that there are 12 possible pairs (g', h') as given. Using [CCN⁺85], we observe that $|C_S(gh)| = 12$ and, moreover, $C_S(gh) \cap C_S(g) = \{1\}$. Thus there exists $x \in C_S(gh)$ such that $g' = g^x, h' = h^x$ and we are done. \square

Note that the condition on elements g', h' given in the proposition is similar to the notion of *rigidity* which has been studied in other contexts (see [Mar10]).

We finish with the promised infinite family.

PROPOSITION 4.7. *Let $x = \frac{2^a + 9}{29}$ where a is a positive integer satisfying $a \equiv 24 \pmod{28}$. Then x is an integer and we set $G = SL_2(8) \times D_{2x}$. Then G is a $(2, 7x, 18)$ -group with associated Euler characteristic -2^{a+1} .*

Proof. It is a trivial matter to check that x is an integer if and only if $a \equiv 24 \pmod{28}$. Let $g_1 \in SL_2(8)$ have order 7 and $h_1 \in SL_2(8)$ have order 9. Checking [CCN⁺85] and applying Proposition 2.5 confirms that we can choose g_1 and h_1 so that they generate $SL_2(8)$ and so that $g_1 h_1$ has order

2. Now choose $g_1 \in D_{2x}$ of order x and g_2 of order 2. Clearly $\langle (g_1, h_1), (g_2, h_2) \rangle = G$. The order of (g_2, h_2) is 18; since 7 never divides $\frac{2^a+9}{29}$ for a a positive integer, it follows that the order of (g_1, h_1) is $7x$, and the order of $(g_1, h_1) \cdot (g_2, h_2)$ is easily observed to be 2.

We calculate the Euler characteristic using (1.2):

$$\chi = 7 \cdot 8 \cdot 9 \cdot 2x \left(\frac{1}{7x} + \frac{1}{18} - \frac{1}{2} \right) = -58x + 18.$$

Now $\chi = -2^{a+1}$ if and only if $x = \frac{2^a+9}{29}$, and we are done. \square

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